Modern Atomic and Optical Physics II

3 hours · 100 points

Answer all questions. This is a closed-book test. A basic scientific calculator is allowed. Constants and formulas are on the last page. Parts I and II need no justification. In Part III, show your reasoning: most of the credit is for the physics. Where an expression shows no $\hbar$, we set $\hbar = 1$, so frequencies stand for energies. The laser detuning is $\delta = \omega_L - \omega_0$ throughout.

(12 points) Part I: True/false questions

Mark each statement True (T) or False (F). Each question is worth 2 points.

1. ( ) A far-detuned laser beam tuned below the atomic resonance ("red detuned", $\delta < 0$) attracts ground-state atoms towards its intensity maximum.
2. ( ) Sub-Doppler polarization-gradient (Sisyphus) cooling works for an atom with a single, non-degenerate ground state.
3. ( ) Spontaneous emission can be suppressed by placing the atom inside a closed cavity that has no mode resonant with the transition.
4. ( ) Collect the resonance fluorescence of a single driven two-level atom into one mode, with no background light. Its intensity correlation function $g^{(2)}(0) = \langle a^\dagger a^\dagger a a \rangle / \langle a^\dagger a \rangle^2$ is zero.
5. ( ) Evaporative cooling can raise the phase-space density even though it removes most of the atoms.
6. ( ) The resonant absorption cross-section of an atom is of the order of its geometric size πa_0^2.

(18 points) Part II: Fill in the blanks

Each question is worth 3 points.

7. The Rb D2 cycling transition has an excited-state lifetime $\tau = 26$ ns. However strongly it is driven, one atom scatters at most _____ photons per second.
8. A ^{87}Rb atom ($m = 1.443 \times 10^{-25}$ kg) moves at 300 m/s. To stop it with a counter-propagating resonant 780 nm beam, it must scatter about _____ photons.
9. A two-level atom is driven on resonance in the strong-driving regime $\Omega \gg \gamma$. Its fluorescence spectrum is the Mollow triplet: a central peak at ω_L and two side peaks at $\omega_L \pm$ _____.
10. For a photon-number state $|n\rangle$ with $n \geq 1$, the correlation function of Question 4 is $g^{(2)}(0) =$ _____.
11. A two-level atom (decay rate γ) sits in a far-detuned laser beam, $|\delta| \gg \Omega, \gamma$. Its ground state feels a light shift U, and it scatters photons at rate Γ_{sc}. In terms of γ and δ, the ratio $\hbar\Gamma_{\text{sc}} / |U| =$ _____.
12. In a 3D harmonic trap of fixed frequencies, the condensation temperature of an ideal Bose gas scales as $T_c \propto N^p$ with the atom number N. The exponent is $p =$ _____.

(70 points) Part III: Problems. Show all your work and justify your answers.

13. (14 points) A two-level atom and its electromagnetic environment. A two-level atom ($|g\rangle$, $|e\rangle$, transition frequency ω_0, dipole matrix element d) decays by spontaneous emission at rate γ. Work in the frame rotating at ω_0, where the undriven atom has $H = 0$.

(a) (4 pts) Use the master equation (formula sheet) with one jump operator $L = \sqrt{\gamma}\sigma$, where $\sigma = |g\rangle\langle e|$. Derive the equations of motion for ρ_{ee} and ρ_{eg}. Show that the population decay time is $T_1 = 1/\gamma$ and the coherence decay time is $T_2 = 2/\gamma$. Now add pure dephasing, with jump operator $L_\varphi = \sqrt{2\gamma_\varphi}\ |e\rangle\langle e|$. Which of T_1, T_2 changes, and how? Explain why $T_2 \le 2T_1$ always.

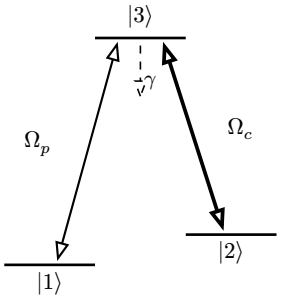
(b) (4 pts) Fermi's golden rule gives $\gamma = \omega_0^3 d^2/(3\pi\varepsilon_0\hbar c^3)$. Explain where the factor ω_0^3 comes from. The Rb D2 line (780 nm) has $\tau = 26$ ns. Estimate the lifetime of an electric-dipole transition with the same d at 6.8 GHz. The real 6.8 GHz hyperfine transition of ^{87}Rb is a magnetic-dipole transition. Its matrix element is smaller by a factor of order $\alpha \approx 1/137$. What does this imply for qubits stored in hyperfine levels?

(c) (6 pts) The atom now sits in a resonant, open optical cavity of mode volume V and quality factor Q. Photons leave the cavity at the energy decay rate $\kappa = \omega_0/Q$. The single-photon coupling is $g = d\sqrt{\omega_0/2\varepsilon_0\hbar V}$. "Open" means the atom still decays into free space at rate γ. Conditioned on no photon being lost, the single-excitation amplitudes of $c_{e0}\ |e,0\rangle + c_{g1}\ |g,1\rangle$ obey, on resonance,

$$\dot{c}_{e0} = -(\gamma/2)c_{e0} - igc_{g1}, \quad \dot{c}_{g1} = -(\kappa/2)c_{g1} - igc_{e0}.$$

(i) Take the bad-cavity limit $\kappa \gg g, \gamma$. Eliminate the cavity adiabatically ($\dot{c}_{g1} \approx 0$). Show that the atom decays at rate $\gamma + 4g^2/\kappa$. Show that the cavity part, divided by γ, is the Purcell factor $F_P = (3/4\pi^2)Q\lambda^3/V$. (ii) Now set $\gamma = \kappa = 0$. Find the eigenstates and energies of the Jaynes–Cummings Hamiltonian $H = g(|e,0\rangle\langle g,1| + |g,1\rangle\langle e,0|)$. Find $P_e(t)$ for an atom that starts in $|e,0\rangle$. For $g \gg \kappa, \gamma$, at what rate does the envelope of $P_e(t)$ decay? Why does the Wigner–Weisskopf picture (Markovian, exponential decay) fail here?

14. (14 points) Electromagnetically induced transparency (EIT). A gas of Λ atoms (figure) has atomic density n_a and length L. A weak probe Ω_p drives $|1\rangle \leftrightarrow |3\rangle$. A strong control field Ω_c drives $|2\rangle \leftrightarrow |3\rangle$. On resonance, the rotating-frame Hamiltonian is $H = -\big(\Omega_p/2\big)|3\rangle\langle 1| - (\Omega_c/2)|3\rangle\langle 2| + \text{h.c.}$ The excited state decays at rate γ. The ground-state coherence ρ_{12} decays at rate $\gamma_{12} \ll \gamma$. Without the control field, the probe intensity falls as $e^{-\mathrm{OD}}$. Here $\mathrm{OD} = n_a\sigma_0 L$ is the optical depth and σ_0 is the resonant cross-section.



(a) (4 pts) Show that $|D\rangle = (\Omega_c\ |1\rangle - \Omega_p\ |2\rangle)/\sqrt{\Omega_c^2 + \Omega_p^2}$ is an eigenstate of H with no amplitude in $|3\rangle$. Explain why an atom in $|D\rangle$ neither absorbs nor scatters light. What role does γ_{12} play?

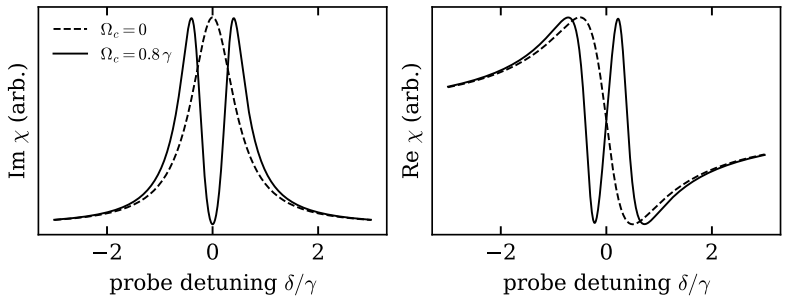
(b) (4 pts) For a weak probe detuned by δ from $|1\rangle \leftrightarrow |3\rangle$ (control still resonant), linear response gives the probe susceptibility

$$k\chi(\delta) = n_a\sigma_0\frac{\gamma}{2}\cdot\frac{i}{\gamma/2 - i\delta + (\Omega_c^2/4)/(\gamma_{12} - i\delta)},$$

with the field obeying $\partial_z\mathcal{E} = i(k/2)\chi\mathcal{E}$. The figure below shows Im χ (absorption) and Re χ (dispersion), control off (dashed) and on (solid). Explain the solid curves near $\delta = 0$: the dip in Im χ, its width, and the slope of Re χ. The refractive index is $n_r = 1 + \text{Re}\ \chi/2$, and the group velocity is $v_g = c/(n_r + \omega\, dn_r/d\omega)$. Show that for $\gamma_{12} \to 0$ and $v_g \ll c$, $v_g \approx \Omega_c^2/(n_a\sigma_0\gamma)$. How does v_g depend on Ω_c and on the density?

(c) (3 pts) A probe pulse of duration $1\,\mu$s enters a medium where $v_g = 17$ m/s. How long is the pulse inside the medium? Where is most of its energy? Explain how slowly turning off Ω_c stores the pulse in the atoms. Why must the switching be adiabatic?

(d) (3 pts) Use (b) to write the delay $\tau_d = L/v_g$ in terms of OD. Expand k Im χ to order δ^2 near $\delta = 0$ (with $\gamma_{12} \to 0$). Show that the transmission window has width $\Delta\omega_{\mathrm{EIT}} \sim \Omega_c^2/\big(\gamma\sqrt{\mathrm{OD}}\big)$. Show that the number of pulse durations that fit into the delay scales as $\sqrt{\mathrm{OD}}$, independent of Ω_c. What is the key resource for an EIT quantum memory?



15. (14 points) Adiabatic passage.

(a) (6 pts) **Rapid adiabatic passage.** A two-level atom has $H = -\delta(t)\ |e\rangle\langle e| + (\Omega/2)(|e\rangle\langle g| + |g\rangle\langle e|)$. The Rabi frequency $\Omega > 0$ is fixed. The detuning is swept as $\delta(t) = \alpha t$, with $\alpha > 0$, from $t = -\infty$ to $+\infty$. The atom starts in $|g\rangle$. Define the mixing angle $\theta \in [0, \pi/2]$ by $\tan 2\theta = \Omega/\delta$. (i) Find the instantaneous eigenvalues $E_\pm$ and eigenstates. Show that following the lower state E_- transfers $|g\rangle \to |e\rangle$. (ii) Adiabatic following requires $|\langle +|\partial_t|-\rangle| \ll E_+ - E_-$. Show that this means $\alpha \ll \Omega^2$. (iii) Why is this transfer insensitive to the exact value of Ω, unlike a π pulse?

(b) (4 pts) **STIRAP** (stimulated Raman adiabatic passage). In the Λ system of Problem 14, with both fields resonant, rename the fields: pump $\Omega_P(t)$ on $|1\rangle \leftrightarrow |3\rangle$ and Stokes $\Omega_S(t)$ on $|2\rangle \leftrightarrow |3\rangle$. The dark state is $|D\rangle = \cos\theta\ |1\rangle - \sin\theta\ |2\rangle$ with $\tan\theta = \Omega_P/\Omega_S$. Show that the sequence "Stokes first, then pump, with overlap" transfers all population from $|1\rangle$ to $|2\rangle$. Describe how P_1, P_2 and P_3 evolve in time. Why must the pulses overlap? Why is this order called counter-intuitive?

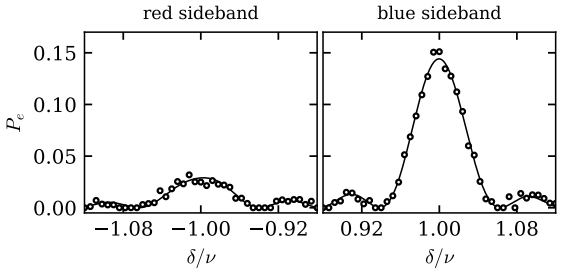
(c) (4 pts) **Errors from decay.** Non-adiabatic coupling mixes in the bright state. This populates $|3\rangle$ with amplitude $|c_3| \sim |\dot{\theta}|\ /\Omega$, where $\Omega = \sqrt{\Omega_P^2 + \Omega_S^2}$. For a transfer of duration τ, estimate the probability of a spontaneous emission from $|3\rangle$. Show that efficient STIRAP requires $\Omega^2\tau \gg \gamma$. Compare with the condition for adiabatic following in a closed system, $\Omega\tau \gg 1$. Which is harder to meet when $\gamma \gg \Omega$? Explain physically why.

16. (14 points) Cooling atoms and ions.

(a) (4 pts) **Doppler limit.** In 1D optical molasses, two counter-propagating beams have detuning $\delta < 0$. Each beam has a low saturation parameter $s = 2\Omega^2/\gamma^2 \ll 1$. Treat spontaneous emission as 1D, so each emitted photon kicks the atom by $\pm\hbar k$. The Langevin description gives a friction force $F = -\beta v$ with $\beta = -8\hbar k^2 s(\delta/\gamma)/\left[1 + (2\delta/\gamma)^2\right]^2$. The momentum diffusion constant, defined by $d\langle p^2\rangle/dt = 2D$, is $D = \hbar^2 k^2 \gamma s/\left[1 + (2\delta/\gamma)^2\right]$. Explain the two physical sources of D. Using $k_B T = D/\beta$, find $T(\delta)$, its minimum, and the detuning where it occurs. Evaluate the minimum for Rb ($\gamma = 2\pi \times 6.1$ MHz).

(b) (6 pts) **Trapped ion.** A two-level ion sits in a harmonic trap of frequency ν. A laser with wave vector k along the trap axis drives it. In the frame rotating at the laser frequency, $H = -\delta\sigma^\dagger\sigma + \nu a^\dagger a + (\Omega/2)\left(\sigma^\dagger e^{ik\hat{x}} + \text{h.c.}\right)$, with $\sigma = |g\rangle\langle e|$. Here $\hat{x} = x_0\left(a + a^\dagger\right)$, $x_0 = \sqrt{\hbar/2m\nu}$, and $\eta = kx_0$ is the Lamb–Dicke parameter. The ion starts in $|g\rangle$. (i) Go to the interaction picture with respect to $-\delta\sigma^\dagger\sigma + \nu a^\dagger a$, and expand to first order in η. Show that there are three resonances, at $\delta = 0, \pm\nu$, with Rabi frequencies Ω, $\eta\sqrt{n}\Omega$ and $\eta\sqrt{n+1}\Omega$. Which is which? (ii) A short, weak probe pulse gives $P_e \propto$ (Rabi frequency)^2. Show that for any motional state with mean phonon number $\overline{n}$, the red-to-blue sideband ratio is $\overline{n}/(\overline{n}+1)$. The figure shows red- and blue-sideband scans taken with such a pulse after sideband cooling. Estimate $\overline{n}$. (iii) Estimate η for $^{40}Ca^+$ at 729 nm with $\nu/2\pi = 1$ MHz.

(c) (4 pts) **Sideband cooling.** The laser is tuned to the red sideband, $\delta = -\nu$. The mean phonon number obeys $\dot{\overline{n}} = -(A_- - A_+)\overline{n} + A_+$, where A_- (A_+) is the rate of transitions that remove (add) a phonon. Find the steady state. Consider only heating by off-resonant excitation of the blue sideband, which is detuned by 2ν. Ignore heating from off-resonant carrier excitation followed by photon recoil. An excitation detuned by Δ has the line shape $(\gamma/2)^2/\left[\Delta^2 + (\gamma/2)^2\right]$. Show that $\overline{n}_{\text{ss}} \sim (\gamma/4\nu)^2$ for $\nu \gg \gamma$. What further condition on η is needed for ground-state cooling, and why?

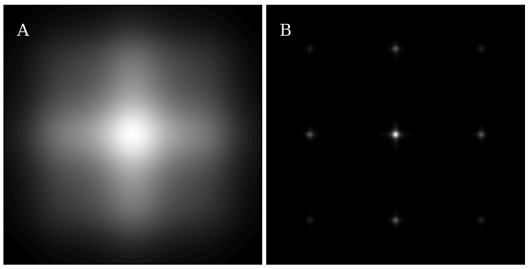


17. (14 points) Bose–Einstein condensates and optical lattices.

(a) (4 pts) At low energy, the interaction between two atoms is set by the s-wave scattering length a. It can be replaced by the pseudopotential $U_0\delta(\boldsymbol{r})$ with $U_0 = 4\pi\hbar^2 a/m$. Explain why one parameter suffices. Take repulsive interactions ($a > 0$) and an isotropic harmonic trap $V(\boldsymbol{r}) = m\omega^2 r^2/2$. Start from the Gross–Pitaevskii equation $\mu\psi = [-\hbar^2\nabla^2/2m + V(\boldsymbol{r}) + U_0\ |\psi|^2]\psi$. Find the density profile when the kinetic term is negligible (Thomas–Fermi approximation). Give the cloud radius R in terms of the chemical potential μ. Normalizing to N atoms gives $\mu = (\hbar\omega/2)(15Na/a_{\mathrm{ho}})^{2/5}$, with $a_{\mathrm{ho}} = \sqrt{\hbar/m\omega}$. Show that neglecting the kinetic term is justified when $Na/a_{\mathrm{ho}} \gg 1$.

(b) (4 pts) **Bogoliubov spectrum.** A uniform condensate of density n_0 has the excitation spectrum $\varepsilon(p) = \sqrt{c_s^2 p^2 + (p^2/2m)^2}$, with sound speed $c_s = \sqrt{U_0 n_0/m}$. Identify the two regimes and the crossover momentum. Use Landau's argument to show that an impurity of mass M moving at speed $v < c_s$ cannot create excitations. Why is an ideal Bose gas not a superfluid? What happens for $a < 0$?

(c) (6 pts) **Optical lattice.** Bosons in a deep 3D cubic lattice of depth V_0 are described by the Bose–Hubbard model $H = -J\sum_{\langle ij\rangle}\left(b_i^\dagger b_j + \text{h.c.}\right) + (U/2)\sum_i n_i(n_i - 1)$. Here $J, U > 0$, and each nearest-neighbour pair $\langle ij\rangle$ is counted once. Each site holds one localized orbital, the Wannier function. (i) How do J and U depend on V_0 (in units of the recoil energy E_r), and why? (ii) Take two sites with two atoms. Find the ground state for $U = 0$ and for $J = 0$. Compute the variance of the atom number on the left site in each case. (iii) Now take a large lattice at one atom per site. Describe the superfluid and Mott-insulator ground states: number fluctuations, phase coherence, excitation gap. In a time-of-flight image, the lattice is switched off and the cloud expands freely. The image then shows the initial momentum distribution $n(\boldsymbol{k}) = |\tilde{w}(\boldsymbol{k})|^2 \sum_{i,j} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} \left\langle b_i^\dagger b_j \right\rangle$, where $\tilde{w}$ is the Fourier transform of the Wannier function. The figure shows the two phases. Which is the superfluid? Explain both images.



Useful constants and formulas

$\hbar = 1.055 \times 10^{-34}$ J s, $h = 6.626 \times 10^{-34}$ J s, $k_B = 1.381 \times 10^{-23}$ J/K, $c = 2.998 \times 10^{8}$ m/s, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $u = 1.661 \times 10^{-27}$ kg, $a_0 = 5.292 \times 10^{-11}$ m.

- Detuning: $\delta = \omega_L - \omega_0$ (laser minus atom).
- Master equation: $\dot\rho = -i[H,\rho] + \sum_k \big(L_k \rho L_k^\dagger - \tfrac{1}{2}\big\{ L_k^\dagger L_k, \rho \big\}\big)$.
- Two-level steady state: $\rho_{ee} = (s/2)/(1 + s + 4\delta^2/\gamma^2)$, with saturation parameter $s = 2\Omega^2/\gamma^2$.
- Propagation of a weak field: $\partial_z \mathcal{E} = i(k/2)\chi\mathcal{E}$; intensity absorption coefficient k Im χ.
- Jaynes–Cummings: $H = \omega_c a^\dagger a + \omega_0 \sigma^\dagger\sigma + g\big(a\sigma^\dagger + a^\dagger\sigma\big)$, $\sigma = |g\rangle\langle e|$, ω_c the cavity frequency.
- Photon momentum $\hbar k = h/\lambda$; recoil energy $E_r = \hbar^2 k^2/2m$.
- Condensation: $n\lambda_{\mathrm{dB}}^3 = 2.612$, $\lambda_{\mathrm{dB}} = h/\sqrt{2\pi m k_B T}$.